

Exact Probability Bounds under Moment-matching Restrictions

Stephen Portnoy¹

April 17, 2015

Abstract

Lindsay and Basak (2000) posed the question of how far from normality could a distribution be if it matches k normal moments. They provided a bound on the maximal difference in c.d.f.'s, and implied that these bounds were attained. It will be shown here that in fact the bound is not attained if the number of even moments matched is odd. An explicit solution is developed as a symmetric distribution with a finite number of mass points when the number of even moments matched is even, and this bound for the even case is shown to hold as an explicit limit for the subsequent odd case.

¹Professor, Department of Statistics, University of Illinois at Urbana-Champaign
corresponding email: sportnoy@illinois.edu

Portnoy (2015) presents results partially correcting claims in Lindsay and Basak (2000) concerning the worst-case approximation of a normal distribution by a distribution that matches a given number of moments. The formal mathematical statements and proofs are given here.

Theorem 1 *Let $\{x_1, \dots, x_n\}$ be any domain of different non-zero values with associated probabilities $\{p_1, \dots, p_n\}$. Suppose the moments are matched*

$$\sum_{i=1}^n p_i x_i^j = M_j \quad j = 1, \dots, n \quad (1)$$

where

$$M_\ell \equiv E Z^\ell, \quad Z \sim \mathcal{N}(0, 1). \quad (2)$$

Then, for $j = 1, \dots, n$,

$$p_j = \frac{\sum_{i=1}^n (-1)^i M_{n-i+1} e_{i-1}(\sim x_j)}{\prod_{i=1}^n (x_i - x_j)} \quad (3)$$

where $e_m(y_1, \dots, y_n)$ denotes the m th elementary symmetric function of its arguments, and the argument $(\sim y_j)$ denotes the $(n-1)$ -vector (y_1, \dots, y_n) with y_j deleted. Furthermore,

$$r \equiv \sum_{j=1}^n p_j = \frac{\sum_{i=1}^n (-1)^i M_{n-i+1} e_{i-1}(x_1, x_2, \dots, x_n)}{\prod_{i=1}^n x_i}. \quad (4)$$

Proof. Consider the Vandermonde matrix

$$V \equiv \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & x_1 & x_2 & \dots & x_n \\ 0 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & x_1^n & x_2^n & \dots & x_n^n \end{pmatrix} \quad (5)$$

Introduce $p_0 \equiv 1 - r = 1 - \sum_{j=1}^n p_j$. Then with p denoting the n -vector with coordinates p_j , equation (1) yields the matrix equations:

$$V \begin{pmatrix} p_0 \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ M \end{pmatrix} \quad V_{22} p = M, \quad (6)$$

where V_{22} is the lower left $n \times n$ submatrix of V and M is the vector of moments. Note also that the argument here does not require M to be the moments of a Normal distribution: any vector will provide the same formulas, though I have no general result providing conditions under which the p_j 's solving (1) need be in $[0, 1]$.

Eisnberg and Fedele (2006) provide a formula for the inverse element of V (where in the notation of that paper, we have taken $x_0 = 0$). Specifically, for $i = 0, 1, \dots, n$:

$$(V^{-1})_{ij} = \phi_{nj} \Psi_{ni}$$

where

$$\phi_{nj} = 1 / \prod_{k=1}^n (x_k - x_j)$$

from the recursion in equation (6) of Eisnberg and Fedele (2006) plus a direct induction argument; and

$$\Psi_{ni} = (-1)^{i+1} e_{n+1-i}(\sim x_j)$$

from equation (26) of that paper.

Thus, noting that $V_{11}^{-1} = 1$, the first row of (6) yields

$$r = \sum_{j=1}^n p_j = \sum_{i=1}^n (-1)^{i+1} M_i e_{n-i}(x_1, \dots, x_n) / \prod_{j=1}^k (x_j - x_0) \quad (7)$$

$$= \sum (-1)^i M_{n-i+1} e_{i-1}(x_1, \dots, x_k) / \prod_{j=1}^n (x_j) . \quad (8)$$

Note that the block-triangular form for V shows that the formula above also gives the corresponding elements of V_{22}^{-1} . Thus, as above, the second row of matrix equation (6) immediately provides (3) . ■

To show that it suffices to consider symmetric discrete distributions,

It is possible to simplify computations by considering only the positive values in a symmetric domain. Specifically, let $\{0 < y_1 < \cdots < y_n\}$ be any domain of positive mass points. Consider the symmetric mixture with (positive) probabilities $p_j/2$ at each of $\pm y_j$ and probability p_0 at zero. Suppose the p_j 's generate m even moments:

$$\sum_{j=1}^n y_j^{2i} = M_{2i}/2 \equiv EZ^{2i}/2 \quad i = 1, \cdots, m .$$

Define

$$r^* \equiv \sum_{j=1}^n p_j . \tag{9}$$

Then, the c.d.f. of the mixture at zero is just $F(0) = 1 - r^*$ and $p_0 = 1 - 2r^*$.

Thus, maximizing the c.d.f. at zero is the same as minimizing r^* .

The following result shows that symmetric distributions suffice. Since the result is not required for the subsequent optimality results, the proof is only sketched.

Theorem 2 *Let $\epsilon > 0$ be given. Let $\{x_1, \cdots, x_n\}$ be any domain for which there are (non-zero) associated probabilities with moments matching k normal moments and whose c.d.f. at zero is within ϵ of the maximal value (over all distributions with k matching moments). Then there is a set of symmetric points, $\{\pm y_j : y_j > 0\}$, with associated positive probabilities also matching k Normal moments and for which the c.d.f. at zero is also within ϵ of its maximal value.*

.

Proof. Choose a perturbation of the domain, $\{x_i^*\}$ so that all $|x_i^*|$ are different and so that the (associated probabilities) p_i^* satisfying the moment

equalities are all positive are such that the c.d.f at zero is within ϵ of the optimum. Introduce the symmetrized domain $\{\{x_i^*\}, \{-x_i^*\}\}$. Consider the linear programming problem given by equations (1) and (2) of Portnoy (2015) applied to the symmetrized domain. Since the original distribution in the statement of the Theorem provides a feasible solution, the linear programming solution provides an optimal value that is also within ϵ of the original optimal value. By symmetry, the solution to the liner programming problem is also symmetric. ■

Theorem 4 (below) requires the derivatives of the probabilities in (3) (with respect to the mass points), which can be computed easily:

Lemma 1 *Consider a set of positive mass points $\{0 < y_1 < y_2 < \dots < y_n\}$. Let p_j be the corresponding probabilities matching moments and let r^* be the sum of probabilities given by (4) (see Theorem 1). Then $\frac{\partial r^*}{\partial y_j}$ alternates in sign for $j = 1, \dots, n$, with the partial derivative with respect to y_1 being positive.*

Proof. For each j , factor $1/y_j$ from r^* :

$$r^* = \frac{1}{y_j} \frac{\sum_{i=1}^k (-1)^i M_{k-i+1} e_{i-1}^k(-y_j)}{2 \prod_{i \neq j} y_i} + g(\sim y_j) \quad (10)$$

$$= p_j^* \frac{\prod_{i \neq j} (y_i - y_j)}{\prod_{i \neq j} y_i} + g(\sim y_j) \quad (11)$$

where the term $g(\sim y_j)$ does not depend on y_j (since each of the numerator terms in g has a factor of y_j that is cancelled by the y_j factor in the denominator product). Since the y_j 's are positive, the product $\prod_{i \neq j} (y_i - y_j)$ alternates in sign, and the theorem follows. ■

Finally, the basic optimality results are presented. First, assume the number of even moments, k , is even, and let $\{y_1, \dots, y_{k/2}\}$ be the squares

of the non-zero roots of the Hermite polynomial $He(2k+1)$. Note that the k non-zero roots of $He(2k+1)$ are located symmetrically about zero, and so there are only $k/2$ squares.

Let M_j denote the normal moments (see (2)). By the standard theory for Gaussian quadrature and symmetry, M_j is given by twice the sum of the $k/2$ even gaussian quadrature weights times the j th power of y_i ($j = 1, \dots, k$). Thus, the weights can be determined by any $k/2$ even moment equalities. Let p_j ($j = 1, \dots, k/2$) satisfy:

$$\sum_{i=1}^{k/2} p_i y_i^{2j} = M_{2j}/2 \quad j = 1, \dots, k/2. \quad (12)$$

That is, the p_j 's are just the even weights. Since these are known to be positive and sum to less than one, they can define a discrete probability distribution by symmetrization and introduction of p_0 . The following theorem shows that this distribution is least favorable in the sense of maximizing p_0 (or, equivalently, maximizing the difference from the normal c.d.f. at zero).

Theorem 3 *If the number of even moments, k , is even, then the solution described above achieves $1/2$ of the bound given in Theorem 2 of Lindsay (2000), and therefore is least favorable.*

Proof.

From (4) the sum of the p_j 's is

$$r^* = (1 - s/M_{2k})/2 \quad (13)$$

where

$$\begin{aligned} s &= \sum_{i=1}^k (-1)^i M_{2(k-i+1)} e_{i-1}^k(y_1, y_2, \dots, y_k) \\ &= \sum_{i=1}^k He[2k+2]_{i+2} M_{2i} \end{aligned}$$

Here, M_ℓ is given by (2) above, and we use the fact that the coefficients of the Hermite polynomial $He[n]$ are just the elementary symmetric functions of the squared roots, $\{y_j\}$ in opposite order.

Let M be the Hankel moment matrix of even moments:

$$M = \begin{pmatrix} 1 & M_2 & M_4 & \cdots & M_{2k} \\ M_2 & M_4 & M_6 & \cdots & M_{2k+2} \\ & \cdots & & & \\ M_{2k} & M_{2k+2} & M_{2k+4} & \cdots & M_{4k} \end{pmatrix}. \quad (14)$$

The upper bound from Lindsay(2000) is $1/M_{1,1}^{-1}$. As noted by Lindsay, by symmetry of the normal distribution, if there is a distribution achieving $.5/M_{1,1}^{-1}$, this distribution must be least favorable (that is, it maximizes the difference from the normal distribution evaluated at zero).

Let C be the matrix whose rows contain zeros and the non-zero coefficients of the Hermite polynomials as follows:

$$\begin{aligned} & He[2k+2]_{(2,4,\dots,2k+2)} \\ & (0, He[2k]_{(2,4,\dots,2k)}) \\ & (He[2k-1]_{(1,3,\dots,2k-1)}, 0) \\ & (0, 0, He[2k-2]_{(2,\dots,2k-2)}) \\ & (0, He[2k-3]_{(1,3,\dots,2k-3)}, 0) \\ & \quad \vdots \\ & (-1, 1, 0, \dots, 0) \end{aligned} \quad (15)$$

Note that

$$\sum_{j=0}^{\ell} M_{2j+2i} He[2\ell+2i+2]_{2j} = \int_x^{2i} \sum x^{2j} He[2\ell+2i+2]_{2j} \varphi(x) dx = 0 \quad (16)$$

by orthogonality of the Hermite polynomials.

Note also that the signs of the coefficients alternate, and $He[2k]_2$ is the first nonzero coefficient and it equals M_{2k} .

Thus, using (16) , $D \equiv MC$ has first row and first column equal to $(D_{11}, 0, 0, \dots, 0)$, where

$$D_{11} = \sum_{i=0}^k He[2k+2]_{i+2} M_{2i} = He[2k+2]_2 - s \quad (17)$$

(from (13)).

That is, MC equals a partitioned block diagonal matrix with a 1×1 and a $(2k-1) \times (2k-1)$ submatrix. Thus, the inverse of D is a similarly partitioned block-diagonal matrix with first row and column

$$(1/D[1, 1], 0, 0, \dots, 0) = (1/(He[2k+2]_2 - s), 0, \dots, 0)$$

Now since $M = DC^{-1}$, $M^{-1} = CD^{-1}$. It follows that the upper $(1, 1)$ element of M^{-1} is

$$He[2k+2]_2 / (He[2k+2]_2 - s) = M_{2k} / (M_{2k} - s)$$

Finally,

$$.5/M^{-1}[1, 1] = .5(M_{2k} - s)/M_{2k} = (1 - s/M_{2k})/2 ,$$

which agrees with r^* (13) . ■

Finally, consider the case when the number of matched even moments is odd.

Theorem 4 *If the number of matched even moments, k , is odd, then there is no distribution matching the k moments that maximizes the difference from the normal c.d.f. at zero. In fact, the maximum difference among moment-matching distributions approaches the maximal value for matching $(k-1)$ even moments, and is the limit through a sequence of discrete mixtures whose maximal mass point, $y_k \rightarrow \infty$.*

Proof.

Assume there is a solution (to the symmetrized problem) with a finite number of mass points y_1, y_2, \dots, y_n . (Note, there is a solution with $n = k$). If $n > k$, the moment equalities determine a manifold of dimension $n - k > 0$, and so it must be possible to move at least one y_j to make p_0 larger (since all partial derivatives are non-zero). Thus, if there is a solution, there is one with k (odd) mass points.

Since all p_j 's are strictly positive for such a solution (in order to match moments), r^* would increase as y_k increases, since the derivative of r^* with respect to y_k is positive by Lemma 1. This contradicts the assumed optimality, and thus proves nonexistence of a solution.

Note that $p_k \leq M_k/y_k^k$ (since M_k is matched by positive values). Thus, p_k must decrease (to zero) as y_k increases. Furthermore, for $j < k$,

$$p_k y_k^j \leq M_k/y_k^{k-j} \rightarrow 0, .$$

Thus the first $k - 1$ moments are nearly determined by $\{y_1, \dots, y_{k-1}\}$, and (since p_k also tends to 0), the optimal p_j^* is also (nearly) determined by the first $(k - 1)$ y_i 's. That is, the optimal p_j^* 's (for case k) are obtained as the limit (as $y_k \rightarrow \infty$) of terms converging to the optimal p_j 's for the $(k - 1)$ case. ■

References

- [1] Eisinberg and Fedele (2006). On the inversion of the Vandermonde matrix, *Applied mathematics and computation*, 174, 1384-1397.

- [2] Lindsay, B. and Basak, P. (2000). Moments determine the tail of a distribution (but not much else), *The American Statistician*, 54, 248-251.
- [3] Portnoy, S. (2015). Maximizing Probability Bounds under Moment-matching Restrictions, *The American Statistician*, 69, 41-44.